# Finite Deflections of Uniformly Loaded, Clamped, Rectangular, Anisotropic Plates

C. Y. CHIA*
*The University of Calgary, Calgary, Alberta, Canada*

## Theme

THE elastic behavior of a rectangular orthotropic plate has been studied by a few authors making use of the von Kármán-type large deflection theory. Yusuff[1] has considered the post-buckling of the plate under edge compression using Fourier series for both deflection and stress function. The large deflection of the plate under lateral load has been treated by Basu and Chapman.[2] Aalami and Chapman[3] have also looked at the problem of the plate under transverse and in-plane loads. In the last two studies the finite-difference technique has been used and the solutions have been restricted to a special class of orthotropic materials.

The present investigation is concerned with the large deflection behavior of a rectangular anisotropic plate with clamped edges. The classical nonlinear theory of elastic plates is applied to the present problem. Hence the classical assumptions for displacements, the strain-displacement relations associated with the von Kármán assumptions, and the equations of equilibrium are the same as in the theory. The material properties or Hooke's law can be introduced at the final stage of the formulation of the governing differential equations. These equations are then solved by the method of perturbation.[4] Because of lack of available solutions for large deflections of anisotropic plates in literature, the present solution is specified for certain special cases and then compared with existing solutions.

## Contents

Let us consider a rectangular plate of length $2a$ in the $x$ direction, width $2b$ in the $y$ direction, and thickness $h$ in the $z$ direction under a uniformly distributed load $q$ per unit area. The origin of the coordinate system is chosen to coincide with the center of the midplane of the undeformed plate. The stress-strain relations for a thin homogeneous anisotropic plate may be written as

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{bmatrix} \tag{1}$$

where $C_{ij}$ is the anisotropic stiffness matrix.

If $u^\circ(x,y)\, v^\circ(x,y)$ and $w^\circ(x,y)$ denote the displacements at the middle surface in the $x$, $y$, $z$ directions, respectively, the governing differential equations expressed in terms of these displacements can be written in the nondimensional form

$$L_1U + L_2V = -W_{,\zeta}L_1W - \lambda W_{,\eta}L_2W \tag{2a}$$

$$L_2U + L_3V = -W_{,\zeta}L_2W - \lambda W_{,\eta}L_3W \tag{2b}$$

$$L_4W = Q + U_{,\zeta}L_5W + (\lambda U_{,\eta} + V_{,\zeta})L_6W + \lambda V_{,\eta}L_7W + \tfrac{1}{2}W_{,\zeta}{}^2L_5W + \lambda W_{,\zeta}W_{,\eta}L_6W + \tfrac{1}{2}\lambda^2W_{,\eta}{}^2L_7W \tag{2c}$$

where the comma denotes partial differentiation with respect to the corresponding coordinate and where

$$\lambda = a/b, \quad \zeta = x/a, \quad \eta = y/b, \quad U = 12au^0/h^2 \tag{3a}$$

$$V = 12av^\circ/h^2, \quad W = 2(3^{1/2})w^0/h, \quad Q = 24(3^{1/2})qa^4/C_{11}h^4 \tag{3b}$$

$$L_1 = (\partial/\partial\zeta^2) + 2\lambda(C_{16}/C_{11})(\partial^2/\partial\zeta\partial\eta) + \lambda^2(C_{66}/C_{11})(\partial^2/\partial\eta^2) \tag{3c}$$

$$L_2 = \frac{C_{16}}{C_{11}}\frac{\partial^2}{\partial\zeta^2} + \lambda\frac{C_{12}+C_{66}}{C_{11}}\frac{\partial^2}{\partial\zeta\partial\eta} + \lambda^2\frac{C_{26}}{C_{11}}\frac{\partial^2}{\partial\eta^2} \tag{3d}$$

$$L_3 = \frac{C_{66}}{C_{11}}\frac{\partial^2}{\partial\zeta^2} + 2\lambda\frac{C_{26}}{C_{11}}\frac{\partial^2}{\partial\zeta\partial\eta} + \lambda^2\frac{C_{22}}{C_{11}}\frac{\partial^2}{\partial\eta^2} \tag{3e}$$

$$L_4 = \frac{\partial^4}{\partial\zeta^4} + 4\lambda\frac{C_{16}}{C_{11}}\frac{\partial^4}{\partial\zeta^3\partial\eta} + 2\lambda^2\frac{C_{12}+2C_{66}}{C_{11}}\frac{\partial^4}{\partial\zeta^2\partial\eta^2} + 4\lambda^3\frac{C_{26}}{C_{11}}\frac{\partial^4}{\partial\zeta\partial\eta^3} + \lambda^4\frac{C_{22}}{C_{11}}\frac{\partial^4}{\partial\eta^4} \tag{3f}$$

$$L_5 = (\partial^2/\partial\zeta^2) + 2\lambda(C_{16}/C_{11})(\partial^2/\partial\zeta\partial\eta) + \lambda^2(C_{12}/C_{11})(\partial^2/\partial\eta^2) \tag{3g}$$

$$L_6 = \frac{C_{16}}{C_{11}}\frac{\partial^2}{\partial\zeta^2} + 2\lambda\frac{C_{66}}{C_{11}}\frac{\partial^2}{\partial\zeta\partial\eta} + \lambda^2\frac{C_{26}}{C_{11}}\frac{\partial^2}{\partial\eta^2} \tag{3h}$$

$$L_7 = \frac{C_{12}}{C_{11}}\frac{\partial^2}{\partial\zeta^2} + 2\lambda\frac{C_{26}}{C_{11}}\frac{\partial^2}{\partial\zeta\partial\eta} + \lambda^2\frac{C_{22}}{C_{11}}\frac{\partial^2}{\partial\eta^2} \tag{3i}$$

If the plate is clamped along its edges, the appropriate boundary conditions are in the nondimensional form

$$U = V = W = W_{,\zeta} = 0 \quad \text{at} \quad \zeta = \pm 1 \tag{4a}$$

$$U = V = W = W_{,\eta} = 0 \quad \text{at} \quad \eta = \pm 1 \tag{4b}$$

Equations (2) are to be solved in conjunction with Eqs. (4). These equations will reduce to those for orthotropic plates by setting $C_{16} = C_{26} = 0$ and to those given by Chien-Yeh[4] and Hooke[5] for isotropic plates by appropriate simplification of the elastic constants.

Let the nondimensional central deflection of the plate, $W(0,0)$, be denoted by $W_0$. The parameters, $Q$, $W$, $U$ and $V$ may be developed into power series with respect to $W_0$ and are assumed to be of the form[4]

$$Q = q_1W_0 + q_3W_0^3 + \cdots \tag{5a}$$

$$W = w_1(\zeta,\eta)W_0 + w_3(\zeta,\eta)W_0^3 + \cdots \tag{5b}$$

$$U = u_2(\zeta,\eta)W_0^2 + u_4(\zeta,\eta)W_0^4 + \cdots \tag{5c}$$

$$V = v_2(\zeta,\eta)W_0^2 + v_4(\zeta,\eta)W_0^4 + \cdots \tag{5d}$$

By definition, the following is required:

$$w_1(0,0) = 1, \qquad w_3(0,0) = w_5(0,0) = \cdots = 0 \tag{6}$$

Introducing Eqs. (5) into Eqs. (2, and 4) and equating like powers of $W_0$, we obtain a series of differential equations and boundary conditions. The first approximation gives by equating the terms in the first power of $W_0$

$$L_4w_1 = q_1 \quad \text{and} \quad w_1 = w_{1,\zeta} = 0 \quad \text{at} \quad \zeta = \pm 1 \tag{7a,b}$$

$$w_1 = w_{1,\eta} = 0 \quad \text{at} \quad \eta = \pm 1 \tag{7c}$$

which are the differential equation and bounary conditions in the nondimensional form for small deflection of clamped anisotropic plates. The second approximation leads to

$$L_1u_2 + L_2v_2 = -w_{1,\zeta}L_1w_1 - \lambda w_{1,\eta}L_2w_1 \tag{8a}$$

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* Associate Professor, Department of Civil Engineering.

$$L_2u_2 + L_3v_2 = -w_{1,\zeta}L_2w_1 - \lambda w_{1,\eta}L_3w_1 \tag{8b}$$

and $u_2 = v_2 = 0$ at $\zeta = \pm 1$ and $\eta = \pm 1$ (8c)

The third approximation yields

$$L_4w_3 = q_3 + u_{2,\zeta}L_5w_1 + (\lambda u_{2,\eta} + v_{2,\zeta})L_6w_1 + \lambda v_{2,\eta}L_7w_1 + \tfrac{1}{2}w_{1,\zeta}{}^2L_5w_1 + \lambda w_{1,\zeta}w_{1,\eta}L_6w_1 + \tfrac{1}{2}\lambda^2 w_{1,\eta}{}^2L_7w_1 \tag{9a}$$

and $w_3 = w_{3,\zeta} = 0$ at $\zeta = \pm 1$, $w_3 = w_{3,\eta} = 0$ at $\eta = \pm 1$ (9b)

The high-order approximation can be obtained similarly but the series solution is, as in Refs. 4 and 5, taken to be

$$Q = q_1W_0 + q_3W_0{}^3, \qquad W = w_1W_0 + w_3W_0{}^3 \tag{10a,b}$$

$$U = u_2W_0^2, \qquad V = v_2W_0^2 \tag{10c,b}$$

Now we seek the solutions of Eqs.(7–9) for a generally orthotropic plate in the form of polynomial

$$w_1 = (1-\zeta^2)^2(1-\eta^2)^2(1 + D_{20}\zeta^2 + D_{11}\zeta\eta + D_{02}\eta^2 + + D_{40}\zeta^4 + D_{31}\zeta^3\eta + D_{22}\zeta^2\eta^2 + D_{13}\zeta\eta^3 + D_{04}\eta^4 + D_{60}\zeta^6 + D_{51}\zeta^5\eta + D_{42}\zeta^4\eta^2 + D_{33}\zeta^3\eta^3 + D_{24}\zeta^2\eta^4 D_{15}\zeta\eta^5 + D_{06}\eta^6) \tag{11a}$$

$$u_2 = (1-\zeta^2)(1-\eta^2)(E_{10}\zeta + E_{01}\eta + E_{30}\zeta^3 + E_{21}\zeta^2\eta + E_{12}\zeta\eta^2 + E_{03}\eta^3 + E_{50}\zeta^5 + E_{41}\zeta^4\eta + E_{32}\zeta^3\eta^2 + E_{23}\zeta^2\eta^3 + E_{14}\zeta\eta^4 + E_{05}\eta^5) \tag{11b}$$

$$v_2 = (1-\zeta^2)(1-\eta^2)(R_{10}\zeta + R_{01}\eta + R_{30}\zeta^3 + R_{21}\zeta^2\eta + R_{12}\zeta\eta^2 + R_{03}\eta^3 + R_{50}\zeta^5 + R_{41}\zeta^4\eta + R_{32}\zeta^3\eta^2 + R_{23}\zeta^2\eta^3 + R_{14}\zeta\eta^4 + R_{05}\eta^5) \tag{11c}$$

$$w_3 = (1-\zeta^2)^2(1-\eta^2)^2(S_{20}\zeta^2 + S_{11}\zeta\eta + S_{02}\eta^2 + S_{40}\zeta^4 + S_{31}\zeta^3\eta + S_{22}\zeta^2\eta^2 + S_{13}\zeta\eta^3 + S_{04}\eta^4 + S_{60}\zeta^6 + S_{51}\zeta^5\eta + S_{42}\zeta^4\eta^2 + S_{33}\zeta^3\eta^3 + S_{24}\zeta^2\eta^4 + S_{15}\zeta\eta^5 + S_{06}\eta^6) \tag{11d}$$

It is seen that Eqs. (11) satisfy their own boundary conditions, Eqs. (6), and the following requirements:

$$w_1(-\zeta,-\eta) = w_1(\zeta,\eta), \qquad w_1(\zeta,-\eta) = w_1(-\zeta,\eta) \tag{12a}$$

$$u_2(0,0) = 0, u_2(-\zeta,-\eta) = -u_2(\zeta,\eta), \qquad u_2(\zeta,-\eta) = -u_2(-\zeta,\eta) \tag{12b}$$

$$v_2(0,0) = 0, v_2(-\zeta,-\eta) = -v_2(\zeta,\eta), \qquad v_2(\zeta,-\eta) = -v_2(-\zeta,\eta) \tag{12c}$$

$$w_3(-\zeta,-\eta) = w_3(\zeta,\eta), \qquad w_3(\zeta,-\eta) = w_3(-\zeta,\eta) \tag{12d}$$

Upon substitution of Eq. (11a) in Eq. (7a), sixteen algebraic equations are generated by equating the constant term to $q_1$ and the terms in $\zeta^2$. $\zeta\eta$, $\eta^2$, $\zeta^4$, $\zeta^3\eta$, $\zeta^2\eta^2$, $\zeta\eta^3$, $\eta^4$, $\zeta^6$, $\zeta^5\eta$, $\zeta^4\eta^2$, $\zeta^3\eta^3$, $\zeta^2\eta^4$, $\zeta\eta^5$, and $\eta^6$ to zero. The fifteen unknowns $D_{ij}$ in Eq. (11a) can be determined by solving the last fifteen equations simultaneously. Introduction of these constants in the first then leads to the solution for $q_1$. The procedure for the determination of coefficients $E_{ij}$, $R_{ij}$ and $S_{ij}$ and the unknown $q_3$ are similar to the above.

It may be observed that these functions can reduce to the solution for orthotropic and isotropic plates by setting

$$D_{11} = D_{31} = D_{13} = D_{51} = D_{33} = D_{15} = 0 \tag{13a}$$

$$E_{01} = E_{21} = E_{03} = E_{41} = E_{23} = E_{05} = 0 \tag{13b}$$

$$R_{10} = R_{30} = R_{12} = R_{50} = R_{32} = R_{14} = 0 \tag{13c}$$

$$S_{11} = S_{31} = S_{13} = S_{51} = S_{33} = S_{15} = 0 \tag{13d}$$

and that the solutions given in Refs. 4 and 5 can be obtained by the following further simplification:

$$D_{60} = D_{42} = D_{24} = D_{06} = S_{60} = S_{42} = S_{24} = S_{06} = 0 \tag{14}$$

Based on the foregoing analysis, the relation between load and central deflection given by Eq. (10a), membrane forces and bending moments (not given here) can be found for any given set of plate properties, aspect ratio and central deflection. The stiffness matrix $D_{ij}$ in Eq. (1) can be calculated from the orthotropic properties with respect to the material principal axes by applying the classical transformation equations. In the case of small deflection of anisotropic plates, Eqs. (10) reduce to $Q = q_1W_0$ and $W = w_1(\zeta,\eta)W_0$ in which $w_1$ is given by Eq. (11a). The numerical result obtained in this work (not presented here) is closer to Whitney's[6] than that obtained by Ashton and Waddoups[7] for the bending moment but, for large degree of anisotropy, the present solution predicts a deflection lower than those given by these authors. In the case of a square isotropic plate, the errors of the present approximation in center deflection, center moment and bending moment at the midpoint of the plate side are 0.02, 1.08, and 0.58%, respectively, in comparison with those given by Timoshenko.[8]

## References


[1] Yusuff, S., "Large Deflection Theory for Orthotropic Rectangular Plates Subjected to Edge Compression," *Journal of Applied Mechanics*, Vol. 19, 1952, pp. 446–450.

[2] Basu, A. K. and Chapman, J. C., "Large Deflexion Behaviour of Transversely Loaded Rectangular Orthotropic Plates," *Proceedings of Institution of Civil Engineers*, Vol. 35, 1966, pp. 79–110.

[3] Aalami, B. and Chapman, J. C., "Large Deflexion Behaviour of Rectangular Orthotropic Plates," *Proceedings of Institution of Civil Engineers*, Vol. 42, 1969, pp. 347–382.

[4] Chien, W. Z. and Yeh, K. Y., "On the Large Deflection of Rectangular Plates," *Proceedings of the 9th International Congress on Applied Mechanics*, Vol. 6, Brussels, 1957, pp. 403–412.

[5] Hooke, R., "Approximate Analysis of the Large Deflection Elastic Behavior of Clamped, Uniformly Loaded, Rectangular Plates," *Journal of Mechanical Engineering Science*, Vol. 2, 1969, pp. 256–268.

[6] Whitney, J. M., "Fourier Analysis of Clamped Anisotropic Plates," *Journal of Applied Mechanics*, Vol. 38, 1971, pp. 530–532.

[7] Ashton, J. E. and Waddoups, M. E., "Analysis of Anisotropic Plates," *Journal of Composite Materials*, Vol. 3, 1969, pp. 148–165.

[8] Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill, New York, 1959, p. 202.